On the Phase Transition Width of K-connectivity in Wireless Multi-hop Networks

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***Abstract*—In this paper, we study the phase transition behavior of k-connectivity ($k = 1, 2, ...$) in wireless multi-hop networks where a total of n nodes are randomly and independently distributed following a uniform distribution in the unit cube $[0, 1]^d$ ($d = 1, 2, 3$), and each node has a uniform transmission range $r(n)$. It has been shown that the phase transition of k-connectivity becomes sharper as the total number of nodes n increases. In this paper we investigate how fast such phase transition happens, and derive a generic analytical formula for the *phase transition width* of k-connectivity for large enough n and for any fixed positive integer k in d-dimensional space by resorting to a Poisson approximation for the node placement. This result also applies to mobile networks where nodes always move randomly and independently. Our simulations show that to achieve a good accuracy, n should be larger than 200 when $k = 1$ and $d = 1$; and n should be larger than 600 when $k \leq 3$ and $d = 2,\ 3$. The results in this paper are important for understanding the phase transition phenomenon; and it also provides valuable insight into the design of wireless multi-hop networks and the understanding of its characteristics.**

***Index Terms*—phase transition width, k-connectivity, connectivity, wireless multi-hop networks, transmission range, average node degree, random geometric graph.**

I. Introduction

Wireless multi-hop networks have been extensively investigated and discussed in recent years. Generally, a wireless multi-hop network, e.g., wireless ad hoc network or wireless sensor network, consists of a group of nodes that communicate with each other over wireless channels [1], [2]. The nodes in such a network operate in a decentralized and self-organized manner and each node can, if needed, act as a router to forward traffic towards its destination [3], [4]. For such wireless multi-hop networks, connectivity (or 1-connectivity) is a prerequisite for providing many network functions [3], [5]. Furthermore, in many applications, k-connectivity for $k > 1$ is required to attain certain properties such as unique localizability [6], [7], robustness in routing [8], fault tolerance [9], [10], etc.

The network is said to be connected (or 1-connected) iff (if and only if) for any pair of two nodes, there is at least one path between them. The network is k-connected iff there is no set of $(k-1)$ nodes whose removal will make the network trivial or disconnect the network. In other words, if any $(k-1)$ nodes fail in the network, the network still remains connected. In this paper, we shall assume a connection model which postulates a transmission range with the property that any two nodes are connected if and only if they are closer than the transmission range. In the context of k-connectivity, for a finite n, it has been shown that there exists a threshold of transmission range above which the network is k-connected with a high probability; and there also exists a threshold of transmission range below which the network is k-connected with a low probability, i.e. the network is more likely not k-connected. The *difference* between the two thresholds defines the so-called *phase transition width*. In the later part of the paper, we will give a more rigorous definition of the phase transition width. Intuitively, the phase transition width gives an indication on how easy/difficult it is to transform a network that is not k-connected into a k-connected network. It has been shown that the phase transition width becomes sharper as the total number of nodes n increases [3], [11], [12], [13]. When n approaches infinity, the phase transition width

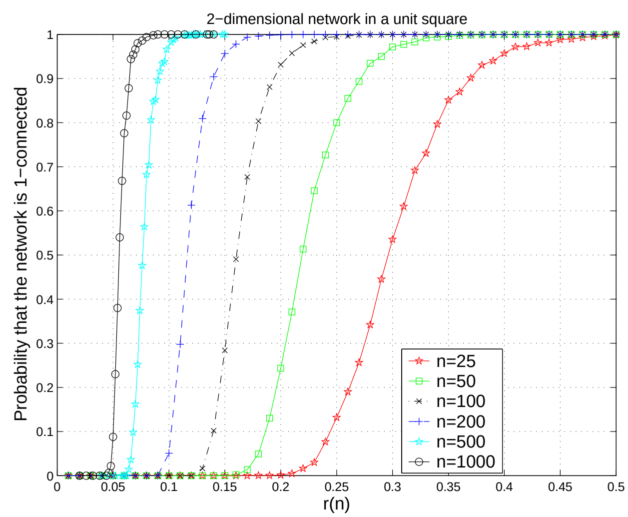

Fig. 1. An illustration of the phase transition behavior of 1-connectivity in 2-dimensional networks through simulations in which a total of n nodes are randomly and uniformly distributed on a unit square. The probability that the network is 1-connected transits from nearly zero to nearly one over a small range of values of the transmission range $r(n)$, and the transition becomes steeper as the number of nodes n increases.

approaches zero and the two aforementioned thresholds will converge to the same value. As an example, Fig. 1 shows the phase transition behavior of 1-connectivity in 2-dimensional networks through simulations. As shown in the figure, when the total number of nodes n is large (e.g., $n = 1000$), it takes a small increase in the transmission range to turn a disconnected network into a connected network. A good understanding of such a phase transition phenomenon is of practical significance for the design of wireless multi-hop

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networks, because it is energy efficient to set the transmission range to be just above the critical threshold for a connected network. If the transmission range is too high, it will waste energy on radio communication and result in interference [14]. For higher dimensional networks (i.e., $d = 2, 3$), all the analytical results hitherto derived on the critical transmission range, at which the network is connected or k-connected, are for an asymptotically *infinite* number of nodes (e.g., [4], [15], [16]). For such a network of an asymptotically infinite number of nodes, the phase transition from not being k-connected to being k-connected occurs at a precise transmission range, and the phase transition width is zero. However, in practice, the total number of nodes in the network is *finite*. So it is crucial to study not just the critical transmission range itself, but the width of the phase transition region, since the phase transition from not being k-connected to being k-connected no longer occurs abruptly.

The central aim of this paper is to investigate analytically how quickly the phase transition of k-connectivity happens in a wireless multi-hop network. We consider that the network is formed in such a way that a total of n nodes are randomly and independently distributed in the unit cube $[0, 1]^d$ ($d = 1, 2, 3$) following a uniform distribution, each node has a uniform transmission range $r(n)$, and any two nodes can communicate with each other iff their Euclidean distance is at most $r(n)$. As an example, Fig. 2 shows an illustration of such network model for different values of d. This model is simple but useful for gaining insights into the operation of wireless multi-hop networks, and it has been widely used by many researchers [3], [4], [5], [6], [11], [15], [17], [18], [19], [20]. In addition, the results obtained using this model can also be extended to other network models. To quantify how fast the phase transition of k-connectivity occurs, we use a metric called *phase transition width*, which is also referred to as *threshold width* in some papers. Let $P_k(n, r(n))$ denote the probability that an instance of a randomly generated network is k-connected. The mapping $r(n) \to P_k(n, r(n))$ is strictly monotonically increasing with $0 < P_k(n, r(n)) < 1$ in some finite interval of $r(n)$, and $P_k(n, r(n)) = 0$ or 1 outside the interval [12]. Let α denote a positive real number. Then define

$$r_k(n, \alpha) := \inf(r > 0 : \; P_k(n, r) \geq \alpha), \; \alpha \in (0, 1). \quad (1)$$

By the definition of $r_k(n, \alpha)$ and the strict monotonicity of $r(n) \to P_k(n, r(n))$, we can readily obtain $P_k(n, r_k(n, \alpha)) = \alpha$. The *phase transition width* over the probability interval $[\alpha, 1 - \alpha]$ of k-connectivity is then defined as

$$\delta_k(n, \alpha) := r_k(n, 1 - \alpha) - r_k(n, \alpha), \; \alpha \in (0, \frac{1}{2}). \quad (2)$$

Henceforth, unless otherwise indicated, the short term phase transition width will be used with α being simply understood. Note that the definition of the phase transition width given by Eq. 2 is closely related to the so-called *finite size scaling* in the physics of percolation and related phenomenon [21], [22].

Previous results derived on the phase transition width of k-connectivity are only for $k = 1$ and $d = 1, 2$, and there is no *generic* result for all $k > 0$ and $d \in \{1, 2, 3\}$. In this paper, we investigate further improvements on the previous results and derive a *generic* analytical formula for the phase transition width $\delta_k(n, \alpha)$ of k-connectivity for large n and for any fixed positive integer $k > 0$ in d-dimensional space ($d = 1, 2, 3$) (Corollary 1). To derive the analytical results, we approximate the node placement (i.e., uniform point process) by a Poisson point process. Such approximation has been widely used in this area, and is shown to be accurate for large n [3], [4], [23], [24], [25]. More details about this Poisson approximation will be given in section V. Based on the result, we then compare the phase transition width for different values of d by fixing k, n and α, which shows that the phase transition width $\delta_k(n, \alpha)$ is larger for higher dimensional networks than that for lower dimensional networks. Similarly we compare the phase transition width for different values of k by fixing d, n and α, which shows that for large n the phase transition width of k-connectivity is approximately the same as the phase transition width of $(k+1)$-connectivity. Alternatively, one may also investigate the phase transition width in terms of the average node degree, i.e. the average number of neighbors per node. This paper also provides an analytical result for the phase transition width measured in the average node degree (Theorem 2). A formal definition of this newly defined phase transition width, denoted as $\delta'_k(n, \alpha)$, will be given in section III. Surprisingly, the newly defined phase transition width turns out to be independent of n and k for large n. We also conduct corresponding simulations to verify our theoretical analysis. Our simulation results show that to achieve a good accuracy, n should be larger than 200 when $k = 1$ and $d = 1$; and n should be larger than 600 when $k \leq 3$ and $d = 2, \; 3$. Our result also applies to mobile networks where nodes always move randomly and independently [10], [26]. To the best of our knowledge, our generic results have never been presented before. These results provide valuable insights into the design and operation of wireless multi-hop networks.

The rest of this paper is organized as follows. Section II briefly reviews related work. Section III describes the models and some basic concepts of graph theory used in the paper. Section IV presents the main results of the paper concerning on the phase transition width of k-connectivity for large n and for any fixed positive integer k in d-dimensional space ($d = 1, 2, 3$). Section V presents proofs for the main results. Section VI presents the simulation results. Finally, Section VII concludes this paper and discuss possible future research directions.

II. Related works

There has been extensive work on the phase transition phenomenon in the past several years. Extensive results have been obtained for Bernoulli random graphs [27]. Usually, a Bernoulli random graph is obtained by randomly distributing n vertices and connecting any pair of two vertices with probability $p(n)$, independently of all other pairs of vertices and the Euclidean distance between the two vertices. Friedgut *et al.* [28] proved that all monotone graph properties[1] have a sharp threshold in a Bernoulli random graph, and the threshold width is $\delta(n, \varepsilon) = O(\log \varepsilon^{-1} / \log n)$. However,

[1]The definition of monotone property is given in Section III.

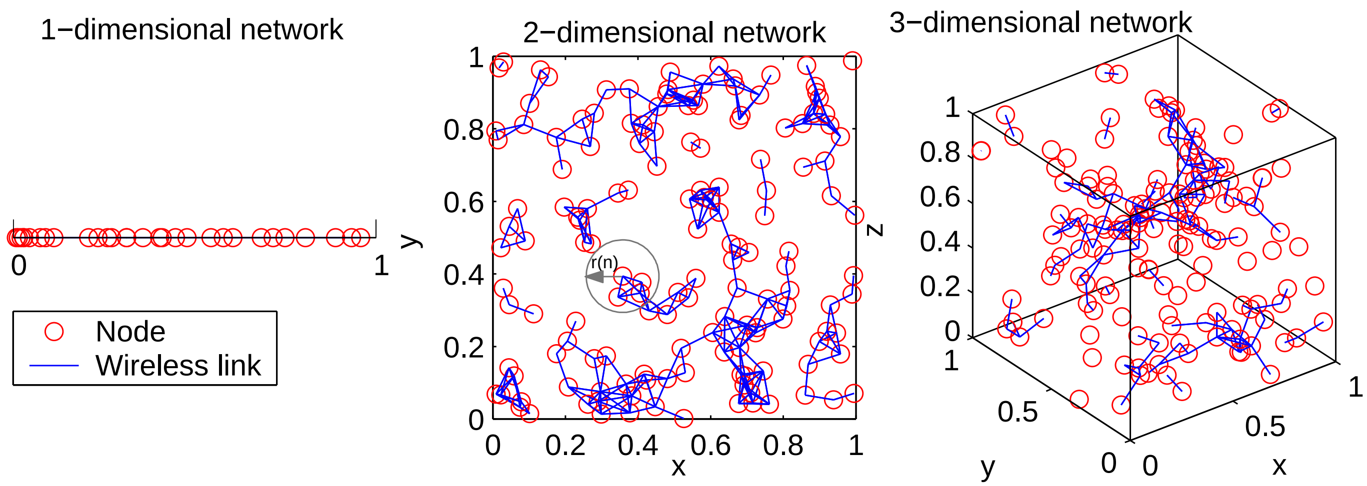


Fig. 2. An illustration of the network model in d-dimensional space ($d = 1, 2, 3$). A total of n nodes are randomly and independently distributed in a unit cube $[0, 1]^d$ following a uniform distribution; each node has a uniform transmission range $r(n)$, and any two nodes can communicate with each other iff their Euclidean distance is at most $r(n)$.

the techniques used for Bernoulli random graphs cannot be applied straightforwardly to random geometric graphs, because in random geometric graphs, the probability of existence of a link between two different nodes is dependent on their Euclidean distance.

In [11], Krishnamachari *et al.* demonstrated the ubiquity of the phase transition phenomenon for monotone graph properties in Bernoulli random graphs and random geometric graphs, and presented some examples in wireless ad hoc networks where a phase transition phenomenon exists, such as connectivity, coordination and probabilistic flooding for route discovery. They then pointed out the significance of understanding phase transitions. In [13], Krishnamachari *et al.* investigated three distributed configuration tasks in wireless multi-hop networks, i.e., partition into coordinating cliques, Hamiltonian cycle formation and conflict-free channel allocation. They showed that these tasks undergo phase transitions with respect to transmission range, and argued that phase transition analysis is useful for quantifying the critical range of energy and bandwidth resources needed for the scalable performance of self-configuring wireless networks. In [4], Ravelomanana showed that the coverage property is subject to abrupt phase transition in 3-dimensional wireless sensor networks. In [6], Aspnes *et al.* exhibited with simulation evidence the phase transition for localizability in wireless sensor networks. And in [29], Raghavan *et al.* investigated phase transition behaviors for the emergence of a giant component in wireless sensor networks and obtained an expression for the critical radius at which the network has a giant component with high probability.

The critical threshold of transmission range was investigated for infinite n in [15], [16] and finite n in [3]. Gupta *et al.* in [15] proved that if the transmission range is set to $r(n) = \sqrt{\frac{\log n + c(n)}{\pi n}}$, the resulting network is asymptotically connected with probability one iff $c(n) \to \infty$ as $n \to \infty$, where the network is formed by uniformly placing n nodes in a unit disc in $\Re^2$. They also conjectured that the network is asymptotically disconnected with probability one iff $c(n) \to -\infty$ as $n \to \infty$. In [16], Wan *et al.* derived a precise critical transmission range at which the network is k-connected ($k > 0$) with probability one as n tends to infinity, where the network is formed by randomly placing n nodes following a uniform distribution in a unit square or disk in $\Re^2$. In [3], Bettstetter investigated the minimum node degree and k-connectivity, where the network is formed by randomly placing n nodes with a uniform distribution in a square of size A in $\Re^2$. Both theoretical results and simulation results showed that k-connectivity undergoes phase transition with respect to transmission range.

In [30], Goel *et al.* proved that all monotone properties in random geometric graph have a sharp threshold. Furthermore, the threshold width for random geometric graphs is much sharper than for Bernoulli random graphs. They showed that for every monotone property, the threshold width $\delta(n, \varepsilon)$ is

$$\delta(n,\varepsilon) = \begin{cases} O(\log^{1/2}\frac{1}{\varepsilon}/n^{1/2}), & d = 1 \\ O(\log^{3/4} n/n^{1/2}), & d = 2 \\ O(\log^{1/d} n/n^{1/d}), & d \geq 3 \end{cases} \tag{3}$$

However, Han *et al.* [12] found that while the results in [30] were derived for a generic monotone property, they may be further sharpened for certain *specific* monotone graph properties such as connectivity. They were seeking to improve the results given by Goel for the property of connectivity in one and two dimensional spaces, and derived the phase transition width for large n, i.e.,

$$\delta_1(n,\varepsilon) = \begin{cases} \frac{C(\varepsilon)}{n} + o(n^{-1}), & d = 1 \\ \frac{C(\varepsilon)}{2}\sqrt{\frac{1}{\pi n \log n}}(1 + o(1)), & d = 2 \end{cases} \tag{4}$$

where $C(\varepsilon) = \log(\frac{\log \varepsilon}{\log(1-\varepsilon)})$. The results are much sharper than the results given in Eq. 3, which indicates that the results in Eq. 3 can be quite conservative for specific monotone properties.

In this paper, we investigate further improvements on Han *et al*'s results. We investigate the phase transition width of k-connectivity for any positive integer k in d-dimensional spaces ($d = 1, 2, 3$). To the best of our knowledge, our generic results have never been presented before.

III. PRELIMINARIES

For many purposes, a wireless multi-hop network can be represented as an undirected graph $G(V, E)$ with a set of vertices V and a set of edges E. Each vertex of the set V uniquely represents a node in the network and each edge of the set E uniquely represents a wireless link in the network, and vice versa. The graph $G(V, E)$ is called the *underlying graph* of the network. In the past several years, the so-called *random geometric graph* has been widely used to represent such wireless multi-hop networks [3], [6], [11], [15], [17], [31], [14]. Throughout this paper, our network model is represented by a random geometric graph $G(n, r(n))$. Typically, a random geometric graph $G(n, r)$ is defined as:

Definition 1 ([18], [23])**.** Given $n \in \mathbb{N}$ and $r \in [0, 1]$, a random geometric graph $G(n, r)$ is a graph in which n vertices are randomly and independently distributed in a unit cube $[0, 1]^d$ ($d = 1, 2, 3, ...$) in $\Re^d$ following a uniform distribution, and any two vertices u and v are directly connected iff $\|u - v\| \leq r$, where $\| \cdot \|$ means the Euclidean norm.

A graph property Λ is a set of undirected and unlabeled graphs. A *monotone graph property* is usually defined as:

Definition 2 ([30], [32])**.** A graph property Λ is increasing iff

$$G \in \Lambda \quad \Rightarrow \quad (\forall G')[(V(G') = V(G) \\ \quad and\ E(G) \subseteq E(G')) \Rightarrow G' \in \Lambda].$$

A graph property Λ is said to be monotone if either Λ or its complement Λ^c is increasing.

The *degree* of a node u, denoted as $deg(u)$, is the number of its neighbors directly connected to it [33]. A node of degree zero is called an *isolated node* (refer to Fig. 3-i). The minimum node degree of a graph G is defined as

$$deg_{min}(G) = \min_{u \in V(G)} \{deg(u)\},$$

and the average node degree of a graph G is

$$C(G) = \frac{1}{n} \sum_{u \in V(G)} deg(u).$$

A graph is said to be k-vertex connected (k-connected for simplicity) iff for any pair of two nodes there exist at least k mutually independent paths connecting them [33], i.e., these paths do not share a common node except for the beginning and the end of the path, (refer to Fig. 3-ii and 3-iii). Equivalently, a graph is k-connected iff there is no set of $(k-1)$ nodes whose deletion will make the network trivial or disconnected. In other words, a k-connected network is able to sustain the failure of $(k - 1)$ nodes, which is a very desirable property for the design of robust routing protocols. A graph is said to be k-edge connected iff for any pair of two nodes there exist at least k mutually edge disjoint paths connecting them. Throughout this paper, we use the term k-connectivity as shorthand for k-vertex connectivity.

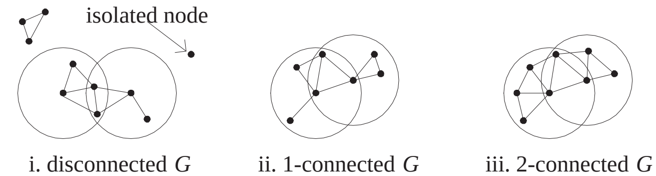


Fig. 3. An illustration of graph k-connectivity. Note that 1-connectivity has the same meaning as connectivity.

In addition to the definition of $\delta_k(n, \alpha)$, we shall define the phase transition width $\delta'_k(n, \alpha)$ measured using the average node degree as follows:

Definition 3. Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), a fixed positive integer $k > 0$ and a real number $\alpha > 0$. Let $C(n)$ denote the average node degree of $G(n, r(n))$, and let $P_k(n, C(n))$ denote the probability that $G(n, r(n))$ is k-connected. Define

$$C_k(n, \alpha) := \inf(C > 0 :\ P_k(n, C) \geq \alpha),\ \alpha \in (0, 1). \quad (5)$$

The new phase transition width $\delta'_k(n, \alpha)$ of k-connectivity in terms of the average node degree is

$$\delta'_k(n, \alpha) := C_k(n, 1 - \alpha) - C_k(n, \alpha),\ \alpha \in (0, \frac{1}{2}). \quad (6)$$

For a random geometric graph $G(n, r(n))$, if the boundary effect is ignored, the average node degree $C(n)$ is given as: $C(n) = 2nr(n)$ for $d = 1$, $C(n) = \pi nr^2(n)$ for $d = 2$ and $C(n) = \frac{4}{3}\pi nr^3(n)$ for $d = 3$ [3], [4]. These easily-derived relations will be used later for relating the two definitions of phase transition width, i.e., $\delta_k(n, \alpha)$ and $\delta'_k(n, \alpha)$.

Throughout the paper, we will use standard mathematical notations concerning the asymptotic behavior of functions, i.e., $f(n) = o(g(n))$ or $f(n) \ll g(n)$ if $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$; $f(n) = O(g(n))$ if there exists a constant c and a value n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$ [34]. Symbols "o" and "O" always apply in the limiting case when $n \to \infty$. To avoid trivialities, we assume n to be sufficiently large. Define the notation $(\cdot)_+$ as $y_+ = y$ if $y \geq 0$ and $y_+ = 0$ if $y \leq 0$. Define π_d ($d = 1, 2, 3$) as $\pi_1 = 1$, $\pi_2 = \pi$ and $\pi_3 = \frac{4}{3}\pi$.

IV. MAIN RESULTS

In this section, we present the main results on the phase transition width $\delta_k(n, \alpha)$ of k-connectivity ($k > 0$) in d-dimensional space ($d = 1, 2, 3$). The proofs are deferred to the next section. For any node u located close to the border of the network area, the coverage area of u, i.e., a sphere centered at u with radius $r(n)$, may be located partially outside the network area. These boundary nodes will have lower average node degree compared with the nodes located in the inner part of the network area. This effect is called the *boundary effect* [25], [35]. Fig. 4 shows an example of the boundary effect in a 2-dimensional network. When $d > 1$, it is complicated to quantify the impact of the boundary effect regarding this specific problem on the probability of k-connectivity, especially for $k > 1$. In this paper, we ignore the boundary effect in our derivation for $d = 2, 3$.

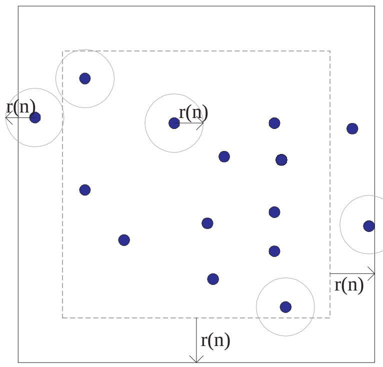


Fig. 4. An illustration of boundary effects in a 2-dimensional network. The nodes located close to the border of the network area have lower average node degree compared with the nodes located in the inner part of the network area.

First, our main result on $r_k(n, \alpha)$ for large n is given in the following:

Theorem 1. *For a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), a fixed positive integer $k > 0$ and a positive real number $\alpha \in (0, 1)$, let $r_k(n, \alpha)$ denote the smallest transmission range at which the probability that $G(n, r(n))$ is k-connected is at least α. Ignore the boundary effect except when $d = 1$. Then, for large n, $r_k(n, \alpha)$ is given by*

$$r_k(n, \alpha) = \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} - \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}}, \quad (7)$$

where

$$F(n,k) := \log n + (k-1)\log(\log n) - \log(k-1)!. \quad (8)$$

Remark. Eq. 7 given in Theorem 1 confirms Theorem 11 in [6]. It is noted in Theorem 11 that given a 2-connected network in $\Re^2$, if we double the transmission range, then the resulting network becomes globally rigid. Eq. 7 yields $2r_2(n, \alpha) > r_6(n, \alpha)$ for all large n. Given that 6-connectivity is a sufficient condition for global rigidity in $\Re^2$ [36], we can obtain that the network with transmission range $2r_2(n, \alpha)$ is globally rigid with high probability. In addition, the transmission range given by Eq. 7 has a similar asymptotic behavior compared with the one given in [5] as n goes to infinity.

Second, the desired result on the phase transition width $\delta_k(n, \alpha)$ for large n is given in the following Corollary 1.

Corollary 1. *Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), a fixed positive integer $k > 0$ and a positive real number $\alpha \in (0, \frac{1}{2})$. Ignore the boundary effect except when $d = 1$. Then, for large n, the phase transition width of k-connectivity $\delta_k(n, \alpha)$ is given by*

$$\delta_k(n, \alpha) = \frac{\log\left(\frac{\log \alpha}{\log(1-\alpha)}\right)}{d\left(\pi_d n\right)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}}(1+o(1)). \quad (9)$$

Remark. Comparing Eq. 4 and Eq. 9, we can see that when $k = 1$ and $d = 1, 2$, Eq. 9 readily reduces to Eq. 4 which appears in [12]. Although we have derived the phase transition width of k-connectivity given by Eq. 9, this result holds only when n is sufficiently large. When n is a small number (especially n is comparable with $k!$), this result does not hold any more. This observation is shown by our proof of Theorem 1 and our later simulation results (there is significant discrepancy between analytical results and simulation results for small n). Also, our simulation results show that n should be larger than 200 when $d = 1$ and $k = 1$; and n should be larger than 600 if $k \le 3$ and $d = 2,\ 3$.

Based on Corollary 1, three further corollaries can also be obtained.

Corollary 2. *Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), a fixed positive integer $k > 0$ and a positive real number $\alpha \in (0, \frac{1}{2})$. Ignore the boundary effect except when $d = 1$. Then, the phase transition width of k-connectivity $\delta_k(n, \alpha)$ and the phase transition width of $(k+1)$-connectivity $\delta_{k+1}(n, \alpha)$ satisfy*

$$\lim_{n\to\infty} \frac{\delta_{k+1}(n, \alpha)}{\delta_k(n, \alpha)} = 1. \quad (10)$$

Remark. Corollary 2 means that for large enough n, $\delta_{k+1}(n, \alpha) \approx \delta_k(n, \alpha)$. In other words, the increase in the transmission range for making the probability that the network is k-connected increase from almost zero to almost one is approximately the same as that required for making the probability that the network is $(k+1)$-connected increase from almost zero to almost one.

Corollary 3. *Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), a fixed positive integer $k > 0$ and a positive real number $\alpha \in (0, \frac{1}{2})$. Ignore the boundary effect except when $d = 1$. Then, for large n, the phase transition width of k-connectivity $\delta_k(n, \alpha)$ in $(j+1)$-dimensional space is larger than that in j-dimensional space, where $j = 1, 2$.*

Remark. Corollary 3 is an easy consequence of the main result (Corollary 1). It indicates that in a higher dimensional network, more transmission power is needed in order to make the probability that the network is k-connected increase from almost zero to almost one.

Corollary 4. *Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), and a fixed positive integer $k > 0$ and a positive real number $\alpha \in (0, \frac{1}{2})$. Ignore the boundary effect except when $d = 1$. Then, for large n, the variation of the phase transition width of k-connectivity $\delta_k(n, \alpha)$ with α is separable from the variation with n, k and d in the sense that for some functions T and Y, there holds $\delta_k(n, \alpha) = T(\alpha)Y(n, k, d)$.*

Remark. Corollary 4 means that if we learn the phase transition width for any $\alpha \in (0, \frac{1}{2})$, it is easy to obtain it for any other $\alpha \in (0, \frac{1}{2})$ given that n, k and d are fixed.

In addition to the results for $\delta_k(n, \alpha)$ in terms of the transmission range, using the similar technique for proving Theorem 1 and Corollary 1, and the relation between the average node degree and the transmission range as shown in section III, we can obtain the following theorem for $\delta'_k(n,\alpha)$ in terms of the average node degree:

Theorem 2. *Consider a random geometric graph $G(n,r(n))$ in $\Re^d$ $(d=1,2,3)$, a fixed positive integer $k>0$ and a positive real number $\alpha\in(0,\frac{1}{2})$. Ignore the boundary effect except when $d=1$. Then, for large n, the phase transition width of k-connectivity $\delta'_k(n,\alpha)$ in terms of the average node degree is given by*

$$\delta'_k(n,\alpha)=\begin{cases} 2\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)+o(1), & d=1;\\ \log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)+o(1), & d\geq 2.\end{cases} \qquad (11)$$

Remark. Theorem 2 indicates that for large enough n, the phase transition width $\delta'_k(n,\alpha)$ measured in the average node degree is only determined by α and is independent of n and k. In addition, the phase transition width $\delta'_k(n,\alpha)$ for $d=2$ and $d=3$ are the same and are the half of that for $d=1$.

V. Proof of the Main Results

In this section, we shall prove the main results given in Section IV. First, we present the following Lemma 1 which we will used below in our proof for Theorem 1.

Lemma 1. *Consider a random geometric graph $G(n,r(n))$ in $\Re^d$ $(d=2,3)$, a fixed positive integer $k>0$ and a real number $\omega\in\Re$, let $\xi(k,n,r(n))$ be the expected number of nodes with degree k. If the boundary effect is ignored, and $r(n)$ is given by*

$$r(n)=r_n(\omega)=\left(\frac{F(n,k)+\omega}{\pi_d n}\right)^{\frac{1}{d}}, \qquad (12)$$

where $F(n,k)$ is as defined in Eq. 8, the following holds:

$$\xi(k-1,n,r_n(\omega)) = e^{-\omega}, \quad k=1. \qquad (13)$$

$$\lim_{n\to\infty}\xi(k-1,n,r_n(\omega)) = e^{-\omega}, \quad k>1. \qquad (14)$$

Proof: As shown in [23, pp. 18], [24, pp. 39], [3], [4], [37], for a set of n nodes, where each node is independently and randomly placed in a finite region of volume V in $\Re^d$, the limiting case obtained by letting $n\to\infty$ and $V\to\infty$ while keeping n/V constant can be regarded as defining a homogeneous Poisson point process of intensity $\rho=n/V$. For large n and large V, i.e., $n\gg 1$ and $V\gg \pi_d r^d(n)$, a homogeneous Poisson point process of intensity $\rho=n/V$ is a close approximation for the uniform distribution. Due to the scaling property of random geometric graphs [23], any realization $G(n,r(n))$ in a *unit* cube in $\Re^d$ coincides with another realization $G(n,\sqrt[d]{V}r(n))$ placed in a cube of volume V in $\Re^d$. Hence, throughout this paper, we focus on $G(n,r(n))$ distributed in a *unit* cube in $\Re^d$, and we assume $n\gg 1$ and $1\gg \pi_d r^d(n)$ so that a homogeneous Poisson point process of intensity $\rho=n/1=n$ can be used to approximate the uniform distribution for the node spatial distribution [18], [23].

Using the Poisson approximation and ignoring the boundary effect, the probability that a particular node i $(i=1,2,...,n)$ has m $(m\geq 0)$ neighbors, denoted as $p_i(m)$, is given by

$$p_i(m)=\frac{\left(n\pi_d r^d(n)\right)^m}{m!}e^{-n\pi_d r^d(n)}, \quad d=2,\ 3. \qquad (15)$$

Therefore, by using the *Palm Theorem* [23, pp. 20, 155] (which captures a form of spatial ergodicity property relating the probabilities that a given node has a certain degree with the expected number of nodes in a network that have a certain degree), and ignoring the boundary effect, the expected number of nodes with degree $(k-1)$, $\xi(k-1,n,r(n))$, is given by

$$\xi(k-1,n,r(n))=n\cdot p_i(k-1)=n\cdot\frac{(n\pi_d r^d(n))^{k-1}}{(k-1)!}e^{-n\pi_d r^d(n)}. \qquad (16)$$

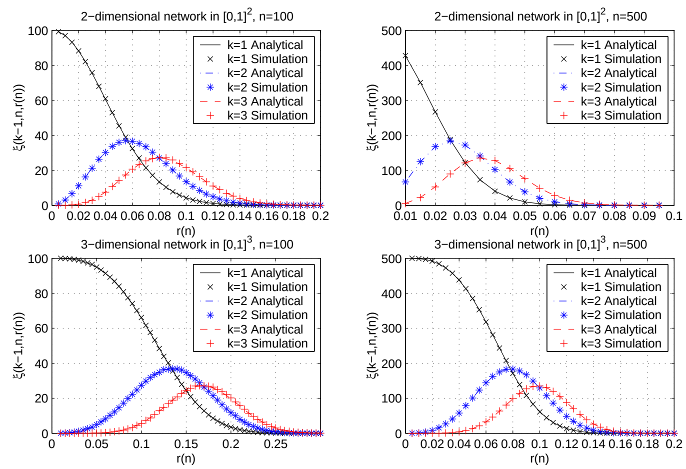


Fig. 5. Expected number of nodes with degree k $(k=1,2,3)$. For the analytical results, the boundary effect is ignored; for the simulation results, the boundary effect is eliminated by using the toroidal distance metric [3].

We also conduct simulations to verify the accuracy of the analytical formula given by Eq. 16. The analytical formula given by Eq. 16 is derived by approximating the uniform point process with a Poisson point process. In Fig. 5, the comparison between the analytical results (Poisson point process approximation) and the simulation results (uniform point process) verifies the accuracy of the Poisson approximation.

For $k=1$, substituting Eq. 12 into Eq. 16, we have

$$\begin{aligned}\xi(0,n,r_n(\omega)) &= n\cdot\frac{(n\pi_d r^d(n))^0}{0!}e^{-n\pi_d r^d(n)}\\ &= n\cdot\exp\left(-n\cdot\pi_d\cdot\frac{\log n+\omega}{n\pi_d}\right)\\ &= e^{-\omega},\end{aligned}$$

which proves Eq. 14.

For $k>1$, substituting Eq. 12 into Eq. 16, we have

$$\begin{aligned}&\lim_{n\to\infty}\xi(k-1,n,r_n(\omega))\\ =\ &\lim_{n\to\infty}\left(n\cdot\frac{(F(n,k)+\omega)^{k-1}}{(k-1)!}\cdot\frac{1}{n}\cdot\frac{(k-1)!}{(\log n)^{k-1}}\cdot e^{-\omega}\right)\\ =\ &e^{-\omega}\lim_{n\to\infty}\left(1+\frac{(k-1)\log(\log n)}{\log n}\right.\\ &\left.-\frac{\log(k-1)!}{\log n}+\frac{\omega}{\log n}\right)^{k-1}\\ =\ &e^{-\omega},\end{aligned}$$

which proves Eq. 14. ∎

A. Proof of Theorem 1

Now, we are able to prove Theorem 1. We first prove the result for $d = 2, 3$ based on Penrose's theorems (i.e., Theorem 1.1, Theorem 2.2) given in [18] and Lemma 1. Since Penrose's theorems in [18] are not valid for $d = 1$, we shall then prove the result for $d = 1$ separately based on Theorem 15 given in [38]. Note that some parts of the proof used here are similar to the arguments used in [12].

*1) **Case when** $d = 2,\ 3$:* For $d = 2,\ 3$, we first introduce two theorems by Penrose that are important for our proof. Let $\gamma_k(n)$ (respectively, $\tau_k(n)$) denote the minimum transmission range at which a random geometric graph $G(n, r(n))$ is k-connected (respectively, has minimum node degree k). Penrose has proved the following two theorems:

Theorem 3 (Theorem 1.1 in [18])**.** *Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d \geq 2$). Given any integer $k > 0$,*

$$\lim_{n\to\infty} Pr\{\gamma_k(n) = \tau_k(n)\} = 1. \tag{17}$$

Theorem 4 (Theorem 1.2 in [18])**.** *Consider a random geometric graph $G(n, r(n))$ in $\Re^d$ ($d \geq 1$). Let $\omega \in \Re$. Given any integer $k > 0$ and $(r_n)_{n\geq 1}$ satisfying the following condition*

$$\lim_{n\to\infty} \xi(k-1, n, r_n) = e^{-\omega}, \tag{18}$$

then it follows that

$$\lim_{n\to\infty} Pr\{\tau_k(n) \leq r_n\} = \exp(-e^{-\omega}). \tag{19}$$

Remark. Theorem 3 above indicates that for very large value of n, k-connectivity is predicted by the minimum node degree. Theorem 4 shows that there is a relation between the proportion of nodes with degree $(k-1)$ and the probability that the network has minimum node degree k. It is also important to notice that Theorem 3 is valid for $d \geq 2$ and Theorem 4 is valid for $d \geq 1$.

For any $\omega \in \Re$ and any positive integer $k > 0$, Theorem 3 and Theorem 4 immediately yield

$$\begin{aligned}&\lim_{n\to\infty} Pr\{\gamma_k(n) \leq r_n(\omega)\}\\ =\ &\lim_{n\to\infty} Pr\{\tau_k(n) \leq r_n(\omega)\} = \exp(-e^{-\omega}).\end{aligned} \tag{20}$$

Hence, Eq. 20 and Lemma 1 yield

$$\lim_{n\to\infty} P_k(n, r_n(\omega)) = \exp(-e^{-\omega}), \tag{21}$$

which plays a *key role* in the proof of Theorem 1.

For each $x \in \Re$, define the $[0,1]$-valued sequence $\{\sigma_n(x), n = 1, 2, 3...\}$ by

$$\sigma_n(x) = \min\left(1, \left(\frac{F(n,k)+x}{\pi_d n}\right)_+^{\frac{1}{d}}\right),\ n = 1, 2, ... \tag{22}$$

Because for any fixed integer $k > 0$ and $x \in \Re$, $\frac{F(n,k)}{n} \to 0$ as $n \to \infty$, there exists a finite integer $N(k, x)$ such that

$$0 < \left(\frac{F(n,k)+x}{\pi_d n}\right)^{\frac{1}{d}} < 1, \quad \forall n > N(k, x).$$

Hence, we have

$$\sigma_n(x) = \left(\frac{F(n,k)+x}{\pi_d n}\right)^{\frac{1}{d}}, \quad \forall n > N(k, x). \tag{23}$$

Therefore, from Lemma 1 and Eq. 21, we have

$$\lim_{n\to\infty} P_k(n, \sigma_n(x)) = \exp(-e^{-x}). \tag{24}$$

Now fix x in $\Re$, from Eq. 24, we can obtain that for each $\varepsilon > 0$, there exists a finite integer $N(\varepsilon, k, x)$ such that

$$\left|P_k(n, \sigma_n(x)) - \exp(-e^{-x})\right| < \varepsilon, \quad \forall n > N(\varepsilon, k, x). \tag{25}$$

It can be easily found that the mapping $\Re \to \Re_+ : x \to \exp(-e^{-x})$ is strictly monotonically increasing and continuous with $\lim_{x\to-\infty} \exp(-e^{-x}) = 0$ and $\lim_{x\to\infty} \exp(-e^{-x}) = 1$. Therefore, for each $\alpha \in (0, 1)$, there exists a unique value of x in $\Re$, denoted as x_α, such that $\exp(-e^{-x_\alpha}) = \alpha$. In fact, from the equality $\exp(-e^{-x_\alpha}) = \alpha$, we have

$$x_\alpha = -\log(-\log \alpha). \tag{26}$$

Hence, fixing x in $\Re$ is equivalent to fixing α in $(0, 1)$. Now fix α in the interval $(0, 1)$, and let ε be sufficiently small such that $0 < 2\varepsilon < \alpha$ and $\alpha + 2\varepsilon < 1$. Then applying Eq. 25 with $x = x_{\alpha+\varepsilon}$ and $x = x_{\alpha-\varepsilon}$ respectively, we have

$$\left|P_k(n, \sigma_n(x_{\alpha+\varepsilon})) - \exp(-e^{-x_{\alpha+\varepsilon}})\right| < \varepsilon,\ \forall n > N(\varepsilon, k, x_{\alpha+\varepsilon}) \tag{27}$$

and

$$\left|P_k(n, \sigma_n(x_{\alpha-\varepsilon})) - \exp(-e^{-x_{\alpha-\varepsilon}})\right| < \varepsilon,\ \forall n > N(\varepsilon, k, x_{\alpha-\varepsilon}). \tag{28}$$

We always assume that n is sufficiently large when necessary. In the rest of this sub-subsection V-A1, we assume that $n > N(\varepsilon, k, \alpha)$ with

$$N(\varepsilon, k, \alpha) = \max\{N(k, x_\alpha),\ N(\varepsilon, k, x_{\alpha+\varepsilon}),\ N(\varepsilon, k, x_{\alpha-\varepsilon})\},$$

where $N(k, x_\alpha)$ represents the finite integer above which Eq. 23 holds.

Since $\exp(-e^{-x_{\alpha\pm\varepsilon}}) = \alpha \pm \varepsilon$, it can be readily obtained from Eq. 27 and Eq. 28 that

$$\alpha < P_k(n, \sigma_n(x_{\alpha+\varepsilon})) < \alpha + 2\varepsilon$$

and

$$\alpha - 2\varepsilon < P_k(n, \sigma_n(x_{\alpha-\varepsilon})) < \alpha.$$

According to the definition of $r_k(n, \alpha)$, we have $P_k(n, r_k(n, \alpha)) = \alpha$. Hence, from the last two inequalities, it follows that

$$P_k(n, \sigma_n(x_{\alpha-\varepsilon})) < P_k(n, r_k(n, \alpha)) < P_k(n, \sigma_n(x_{\alpha+\varepsilon})).$$

Because of the strict monotonicity of the map $r(n) \to P_k(n, r(n))$, we have

$$\sigma_n(x_{\alpha-\varepsilon}) < r_k(n, \alpha) < \sigma_n(x_{\alpha+\varepsilon}). \tag{29}$$

Define $\eta(n, \alpha) := r_k(n, \alpha) - \sigma_n(x_\alpha)$, then it can be obtained from Eq. 29 that

$$\sigma_n(x_{\alpha-\varepsilon}) - \sigma_n(x_\alpha) < \eta(n, \alpha) < \sigma_n(x_{\alpha+\varepsilon}) - \sigma_n(x_\alpha). \tag{30}$$

For any fixed $k > 0$ and $x \in \Re$, it is true that

$$\begin{aligned} & \lim_{n\to\infty} \frac{x}{F(n,k)} \\ = & \lim_{n\to\infty} \frac{x}{\log n + (k-1)\log(\log n) - \log(k-1)!} = 0. \end{aligned}$$

Hence, from Eq. 23, we have

$$\begin{aligned} \sigma_n(x) &= \left(\frac{F(n,k)+x}{\pi_d n}\right)^{\frac{1}{d}} \\ &= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{x}{F(n,k)}\right)^{\frac{1}{d}} \\ &= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{1}{d}\frac{x(1+o(1))}{F(n,k)}\right), as\ n \to \infty. \end{aligned}$$

Therefore, for any fixed $k > 0$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned} & \sigma_n(x_{\alpha\pm\varepsilon}) - \sigma_n(x_\alpha) \\ = & \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{1}{d}\frac{x_{\alpha\pm\varepsilon}(1+o(1))}{F(n,k)}\right) \\ & -\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{1}{d}\frac{x_{\alpha}(1+o(1))}{F(n,k)}\right) \\ = & \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \frac{x_{\alpha\pm\varepsilon} - x_\alpha}{F(n,k)d}(1+o(1)), as\ n \to \infty \end{aligned} \tag{31}$$

Because Eq. 30 holds for all $n > N(\varepsilon, k, \alpha)$, it must be valid when $n \to \infty$ as well. And as $n \to \infty$, the small order part $o(1)$ in Eq. 31 goes to zero. Hence, from Eq. 30 and Eq. 31, we have

$$\begin{aligned} x_{\alpha-\varepsilon} - x_\alpha &\le \liminf_{n\to\infty} \left(\frac{F(n,k)d(1+o(1))}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right) \\ &= \liminf_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right) \end{aligned}$$

and

$$\begin{aligned} x_{\alpha+\varepsilon} - x_\alpha &\ge \limsup_{n\to\infty} \left(\frac{F(n,k)d(1+o(1))}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right) \\ &= \limsup_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right). \end{aligned}$$

Because ε can be chosen to be arbitrarily small, and as stated earlier $x_\alpha = -\log(-\log\alpha)$ is a continuous and strictly monotonically increasing function of α for $\alpha \in (0, 1)$, it can be shown that

$$\lim_{\varepsilon\downarrow 0}(x_{\alpha-\varepsilon} - x_\alpha) = \lim_{\varepsilon\downarrow 0}(x_{\alpha+\varepsilon} - x_\alpha) = 0.$$

Hence, we have

$$\begin{aligned} & \liminf_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right) \\ = & \limsup_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right) = 0. \end{aligned}$$

Thus, given that

$$\lim_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}\eta(n,\alpha)\right) = 0$$

holds, it must be true that

$$\eta(n,\alpha) = o\left(\frac{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}{F(n,k)d}\right) = o\left(\frac{1}{d\,(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}\right).$$

Hence, we have

$$\begin{aligned} & r_k(n,\alpha) = \sigma_n(x_\alpha) + \eta(n,\alpha) \\ = & \left(\frac{F(n,k)+x_\alpha}{\pi_d n}\right)^{\frac{1}{d}} + o\left(\frac{1}{d\,(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}\right) \\ = & \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{x_\alpha(1+o(1))}{F(n,k)d}\right) \\ & + o\left(\frac{1}{d\,(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}\right) \\ = & \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} - \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}. \end{aligned}$$

The proof of Theorem 1 for $d = 2, 3$ is complete.

*2) **Case when** $d = 1$:* For $d = 1$, when $k = 1$, Eq. 7 readily becomes Han *et al*'s result (see Eq. 4 in [12]), therefore, the result given in Theorem 1 is true for $d = 1$ and $k = 1$.

When $k > 1$, we shall prove the result based on Theorem 15 given in [38]. It is noted in Theorem 15 in [38] that given a random geometric graph $G(n, r(n))$ in 1-dimensional space and $\omega \in \Re$, for any positive integer $k > 1$, if $r(n)$ is given by

$$r(n) = \frac{1}{n}(\log n + (k-1)\log(\log n) - \log(k-1)! + \omega),$$

then

$$\lim_{n\to\infty} P_k(n, r(n)) = \exp(-e^{-\omega}).$$

Based on the critical radius given in Theorem 15 of [38] and using the same technique as described in sub-subsection V-A1, we can obtain

$$r_k(n,\alpha) = \frac{F(n,k)}{n} - \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{n},\ d = 1,$$

which agrees with the result given in Theorem 1 for $d = 1$ and $k > 1$. The proof of Theorem 1 for $d = 1$ is complete. It is important to notice that the boundary effect affects neither the derivation of Han *et al*'s result in [12] nor the derivation of Theorem 15 in [38]. Hence, our result in this paper is not affected by the boundary effect when $d=1$.

Combining sub-subsections V-A1 and V-A2, we have finally proved Theorem 1.

B. Proof of Corollary 1

Based on Theorem 1, for any fixed positive integer $k>0$ and $d \in \{1,2,3\}$, the phase transition width $\delta_k(n,\alpha)$ for large n given in Corollary 1 can readily be derived as

$$\begin{aligned}
\delta_k(n,\alpha) &= r_k(n,1-\alpha) - r_k(n,\alpha) \\
&= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} - \frac{\log\left(\log\left(\frac{1}{1-\alpha}\right)\right)(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}} \\
&\quad - \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} + \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}} \\
&= \frac{\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)}{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}}(1+o(1)).
\end{aligned}$$

The proof of Corollary 1 is complete.

C. Proof of Corollary 2

Based on Corollary 1, for any fixed positive integer $k>0$ and $d \in \{1,2,3\}$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\delta_{k+1}(n,\alpha)}{\delta_k(n,\alpha)} \\
&= \lim_{n\to\infty}\left(\frac{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k+1)\right)^{\frac{d-1}{d}}(1+o(1))}\right) \\
&= \lim_{n\to\infty}\left(\frac{\log n + (k-1)\log(\log n) - \log(k-1)!}{\log n + k\log(\log n) - \log k!}\right)^{\frac{d-1}{d}} \\
&= \lim_{n\to\infty}\left(\frac{1+\frac{(k-1)\log(\log n)}{\log n} - \frac{\log(k-1)!}{\log n}}{1+\frac{k\log(\log n)}{\log n} - \frac{\log k!}{\log n}}\right)^{\frac{d-1}{d}} \\
&= 1.
\end{aligned}$$

The proof of Corollary 2 is complete. From Corollary 2, we can obtain that $\delta_{k+1}(n,\alpha) \approx \delta_k(n,\alpha)$ for large enough n.

As an example, Fig. 6 shows the analytical results for the phase transition width of k-connectivity ($k=1,2,3$) in 2-dimensional space, which are calculated from Eq. 9 (set $d=2$ and $k=1,2,3$). α is set two typical values, i.e., 0.4 (close to 0.5) and 0.01 (close to 0). n is set up to a large value (i.e.,10000) so that the way in which $\delta_k(n,\alpha)$ varies with n can be observed. Note that in the calculation, we omit the small order part, i.e., $o(1)$ in $(1+o(1))$.

D. Proof of Corollary 3

Let $\delta_k^{(d)}(n,\alpha)$ denote the phase transition width of k-connectivity in d-dimensional space as we want to emphasize the dependence of $\delta_k(n,\alpha)$ on d. Let j be either 1 or 2. Then,

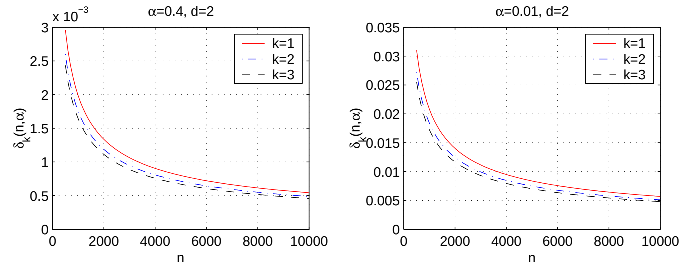


Fig. 6. Analytical results for the phase transition width of k-connectivity ($k=1,2,3$) in 2-dimensional space. The value of α is fixed in each scenario.

based on Corollary 1, we have

$$\begin{aligned}
\frac{\delta_k^{(j+1)}(n,\alpha)}{\delta_k^{(j)}(n,\alpha)} &= \frac{\frac{\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)(1+o(1))}{(j+1)(\pi_{j+1}n)^{\frac{1}{j+1}}(F(n,k))^{\frac{j}{j+1}}}}{\frac{\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)(1+o(1))}{j(\pi_j n)^{\frac{1}{j}}(F(n,k))^{\frac{j-1}{j}}}} \\
&= \frac{j}{j+1}\frac{(\pi_j)^{\frac{1}{j}}}{(\pi_{j+1})^{\frac{1}{j+1}}}\left(\frac{n}{F(n,k)}\right)^{\frac{1}{j(j+1)}}(1+o(1)). \quad (32)
\end{aligned}$$

For any fixed positive integer $k>0$ and j (either $j=1$ or $j=2$), there exists a finite integer $N(k,j)$ such that

$$\frac{j}{j+1}\frac{(\pi_j)^{\frac{1}{j}}}{(\pi_{j+1})^{\frac{1}{j+1}}}\left(\frac{n}{F(n,k)}\right)^{\frac{1}{j(j+1)}}(1+o(1)) > 1$$

for all $n > N(k,j)$. Hence, from Eq. 32, we have $\delta_k^{(j+1)}(n,\alpha) > \delta_k^{(j)}(n,\alpha)$ ($j=1,2$) for large n. The proof of Corollary 3 is complete.

As an example, Fig. 7 shows the analytical results for the phase transition width of 1-connectivity in d-dimensional space ($d=1,2,3$), which are calculated using Eq. 9 ($k=1$ and $d=1,2,3$). Other settings are the same as in Fig. 6. In the calculation, we still omit the small order part $o(1)$ in $(1+o(1))$. We can see that $\delta_k^{(3)}(n,\alpha) > \delta_k^{(2)}(n,\alpha) > \delta_k^{(1)}(n,\alpha)$ when n is larger than a certain threshold.

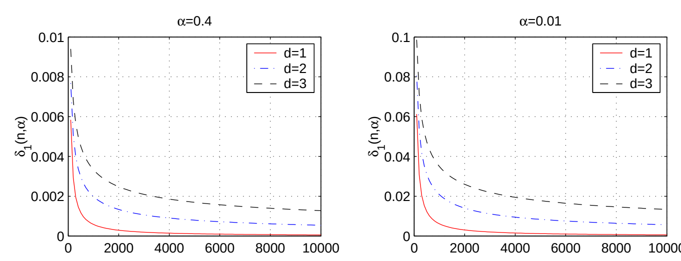


Fig. 7. Analytical results for the phase transition width of 1-connectivity in d-dimensional space ($d=1,2,3$). The value of α is fixed in each scenario.

E. Proof of Corollary 4

Let $\delta_k^{(d)}(n,\alpha)$ denote the phase transition width of k-connectivity in d-dimensional space as we want to emphasize the dependence of $\delta_k(n,\alpha)$ on d. We assume that n is large enough such that the small order part $o(1)$ in Eq. 9 can be ignored. Then, for any fixed n, k and d, we have

$$\begin{aligned}\delta_k^{(d)}(n,\alpha) &\approx \frac{\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)}{d\left(\pi_d n\right)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}} \\ &= \log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)\cdot\frac{1}{d\left(\pi_d n\right)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}} \\ &= T(\alpha)\cdot Y(n,k,d).\end{aligned}$$

It is obvious that for any fixed n, k and d, the term $Y(n,k,d)$ is fixed. In addition, the term $Y(n,k,d)$ is independent of α. Therefore, the value of $Y(n,k,d)$ does not change as α varies given that n, k and d are fixed. If we know $\delta_k^{(d)}(n,\alpha)$ for any $\alpha \in (0,\frac{1}{2})$, we can derive the value of $Y(n,k,d)$, which leads us to derive $\delta_k^{(d)}(n,\alpha)$ for any other $\alpha \in (0,\frac{1}{2})$ with n, k and d fixed. The proof of Corollary 4 is complete.

As an example, Fig. 8 shows the dependence of the phase transition width of k-connectivity on α in d-dimensional space ($d = 1,2,3$). These analytical results are calculated from Eq. 9, and the small order part $o(1)$ in $(1+o(1))$ is omitted. We can see that the variation of $\delta_k^{(d)}(n,\alpha)$ with α has the same functional dependence irrespective of n, k and d, save for a scaling constant defined by these latter variables.

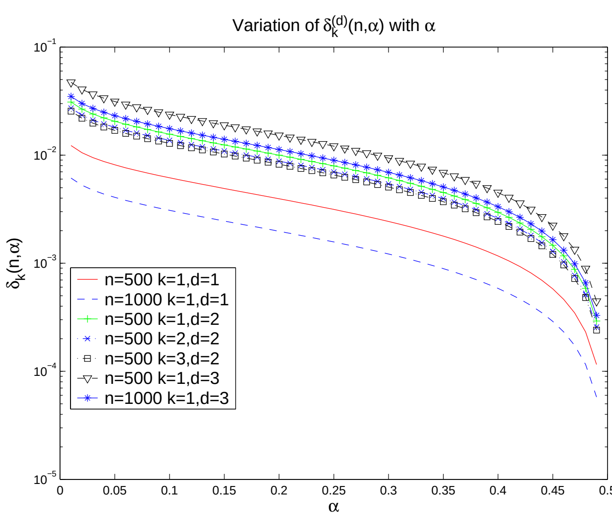


Fig. 8. Analytical results for the phase transition width of k-connectivity *versus* α in d-dimensional space $d = 1,2,3$.

F. *Proof of Theorem 2*

For $d = 2,3$, ignore the boundary effect, the average node degree is $n\pi_d r^d(n)$. In the same way as shown in the proof of Theorem 1 and Corollary 1, we can readily obtain result given in Eq. 11 for $d = 2,3$.

For $d = 1$, the average node degree is $2nr(n)$. Again, using the same method as shown in the proof of Theorem 1 and Corollary 1, the result follows.

VI. Simulations

We have theoretically investigated the phase transition width of k-connectivity $\delta_k(n,\alpha)$ for any positive integer k and for large n in d-dimensional space ($d = 1,2,3$) in Sections IV and V. In this section, we conduct simulations to verify our theoretical analysis.

A. *Simulation deployment*

In the simulations, we consider that a total of n nodes are randomly and independently distributed in a unit cube $[0,1]^d$ ($d = 1,2,3$) according to a uniform distribution. All the nodes have the same transmission range $r(n)$. We programmed a tool in *C++* for computing the phase transition width of k-connectivity $\delta_k(n,\alpha)$ for $k = 1,2,3$. Simulations become very computationally intensive and time consuming for $k > 3$ and large values of n. Therefore we limited k to 3 and n to 1500 in the simulations.

B. *Computing the phase transition width $\delta_k(n,\alpha)$*

Here we give a brief description of the determination of the phase transition width of k-connectivity $\delta_k(n,\alpha)$ in simulations. The following are the main steps:

1) For any given n, distribute n nodes randomly and independently in a unit cube $[0,1]^d$ ($d = 1,2,3$) following a uniform distribution. Then we obtain a network topology Γ_i ($i = 1,2,3,...$). For this network topology Γ_i and each $k \in \{1,2,3\}$, find the corresponding minimum transmission range $r_k(i)$ which makes the network k-connected.
2) Repeat step 1) for a large number of times N (e.g., $N = 10000$), then we obtain a set of N random topologies $\{\Gamma_i, i = 1,2,3,...,N\}$, and three sets of N corresponding minimum transmission ranges $\{r_k(i), i = 1,2,3,...,N\}$ for $k = 1,\ 2$ and 3 respectively, where each set contains N transmission range values.
3) Reorder each set of the N transmission ranges (i.e., $\{r_k(i), i = 1,2,3,...,N\}$ for $k = 1$ or 2 or 3) in an ascending order, such that $r'_k(i) \le r'_k(i+1)$ in the new ordered sets for all $i \in [1, N-1]$.
4) For each $k \in \{1,2,3\}$, let $j = \lceil N \times \alpha \rceil$ and $l = \lceil N \times (1-\alpha) \rceil$. Then the j-th (respectively, l-th) item $r'_k(j)$ (respectively, $r'_k(l)$) in the new ordered set is approximately the minimum transmission range at which the network is k-connected with probability α (respectively, $(1-\alpha)$). Finally, the difference between these two transmission ranges $\tilde{\delta}_k(n,\alpha) = r'_k(l) - r'_k(j)$ is approximately the phase transition width over the probability interval $[\alpha, 1-\alpha]$ of k-connectivity. The larger N is, the more accurate the computed phase transition width is. However, a large N will cost a large amount of time.
5) Repeat steps 1) to 4) for different values of n to obtain the phase transition width of k-connectivity for different values of n.

C. *Eliminating the boundary effect*

An important aspect in the simulation is to eliminate the boundary effect. As we know, the simulation is performed in a bounded area (e.g., unit cube $[0,1]^d$), nodes located at the edges and borders of the network area only have links toward the middle of the network area. Hence, their average node degree is lower than that of the nodes located in the middle of the network area. Since the analytical results are derived without considering the boundary effect, the boundary effect in the simulation will make it impossible to compare the simulation results with the analytical results.

Usually, there are two ways to avoid the boundary effect. The first one is to divide the entire simulation area into two disjoint subareas [37], [39]: a boundary subarea Z_{out} with a width of at least $r(n)$, and an inner subarea Z_{in}. Fig. 9 shows two examples for 2-dimensional networks. Only nodes

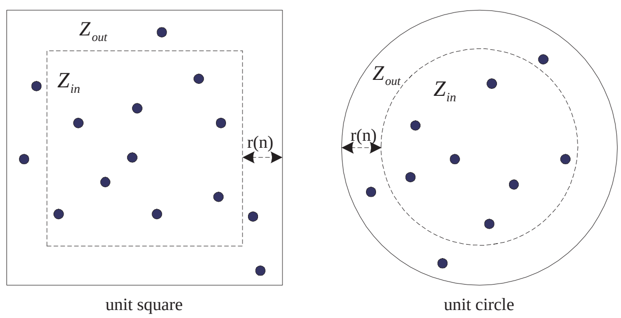


Fig. 9. An illustration of the first method for avoiding the boundary effect in 2-dimensional network area. Only nodes located in subarea Z_{in} are counted for the statistics in the simulation.

that are located in the inner subarea Z_{in} are counted for the statistics of the simulations. In fact, the nodes in the boundary subarea Z_{out} can be used as relay nodes for building links with the nodes located in the inner subarea Z_{in}, but they are not counted for the statistics. One of the disadvantages of this technique is that the number of nodes located in the inner subarea Z_{in} decreases as $r(n)$ increases when the size of the network area is fixed [3].

The second approach to avoid the boundary effect is to use the *toroidal distance metric* [3], [40]. The principle of this technique is to model the network in such a way that nodes at the border are allowed to have links to the nodes at the opposite border. For example, nodes at the right border can have links to the nodes at the left border, and in the same way, nodes at the bottom border can have links to the nodes at the top border. In such a way, each node has the same average node degree and the boundary effect is eliminated. In this paper, we use the second method, i.e., toroidal distance metric, to avoid the boundary effect in simulations.

D. Simulation results

Fig. 10 shows the analytical results and the simulation results for the phase transition width of 1-connectivity $\delta_1(n, \alpha)$ in d-dimensional space $d = 1, 2, 3$. The value of n is varied between 100 and 1500, α is set two typical values, i.e., 0.4 (close to 0.5) and 0.05 (close to 0). For the analytical results, the boundary effect is ignored; when calculating the analytical results by Eq. 9, the small order part is omitted, i.e., $o(1)$ in the term $(1 + o(1))$ is ignored. For the simulation results, the boundary effect is eliminated by using the toroidal distance metric in order to have a fair comparison with the analytical results. We can see that for each $d \in \{1, 2, 3\}$, the phase transition width of 1-connectivity decreases as n grows. It is obvious that there is significant discrepancy between the analytical results and the simulation results for small values of n (e.g., $n < 200$). This is because the small order part $o(1)$ in the analytical results is significant when n is small, especially when n is comparable with $k!$. However, the small order part $o(1)$ goes to zero as n goes to infinity. So the discrepancy decreases as n increases. We can see that although there is significant discrepancy when n is not large enough, the simulation results have the same decreasing property as the analytical results. We can also see that the phase transition width of 1-connectivity is larger for $d = 3$ than that for $d = 2$, and similarly, the phase transition width is larger for $d = 2$ than that for $d = 1$, which verifies Corollary 3. This means that in a higher dimensional network, more transmission power is needed in order to make the probability that the network is k-connected transit from almost zero to almost one.

Fig. 11 shows the analytical results and the simulation results for the phase transition width of k-connectivity $\delta_k(n, \alpha)$ ($k = 1, 2, 3$) in 2-dimensional space. Other settings are the same as in Fig. 10. We can see that when $d = 2$, the phase transition width of k-connectivity decreases as n increases. The figure also indicates that the difference between $\delta_k(n, \alpha)$ and $\delta_{k+1}(n, \alpha)$ becomes smaller as n gets larger. It means that $\delta_k(n, \alpha) \approx \delta_{k+1}(n, \alpha)$ when n is large enough, which is consistent with Corollary 2. In other words, the energy required for making the probability that the network is k-connected increase from almost zero to almost one is approximately the same as the energy required for making the probability that the network is $(k+1)$-connected increase from almost zero to almost one.

Fig. 12 shows the dependence of the phase transition width of k-connectivity on α given that n, k and d are fixed. We can see that the variation of $\delta_k(n, \alpha)$ with α has the same functional dependence irrespective of n, k and d. This verifies Corollary 4. We can also see that for fixed n, k and d, $\delta_k(n, \alpha)$ decreases as α increases. In addition, the discrepancy between the analytical results and the simulation results becomes significant when α is very small. The reason for this is that the small order part $o(1)$ in the analytical results becomes significant when α is very close to zero.

Simulation results also suggested that in order to achieve a good accuracy, n should be larger than 200 when $k = 1$ and $d = 1$; and n should be larger than 600 when $k \leq 3$ and $d = 2,\ 3$.

VII. Conclusion and future work

In this paper, we investigated the phase transition behavior of k-connectivity with respect to the transmission range of nodes in wireless multi-hop networks, where n nodes are randomly and uniformly distributed in a unit cube $[0, 1]^d$, and each node has a uniform transmission range $r(n)$. The phase transition behavior is associated with the transmission range, thus the transmission power of nodes. It is desirable to control

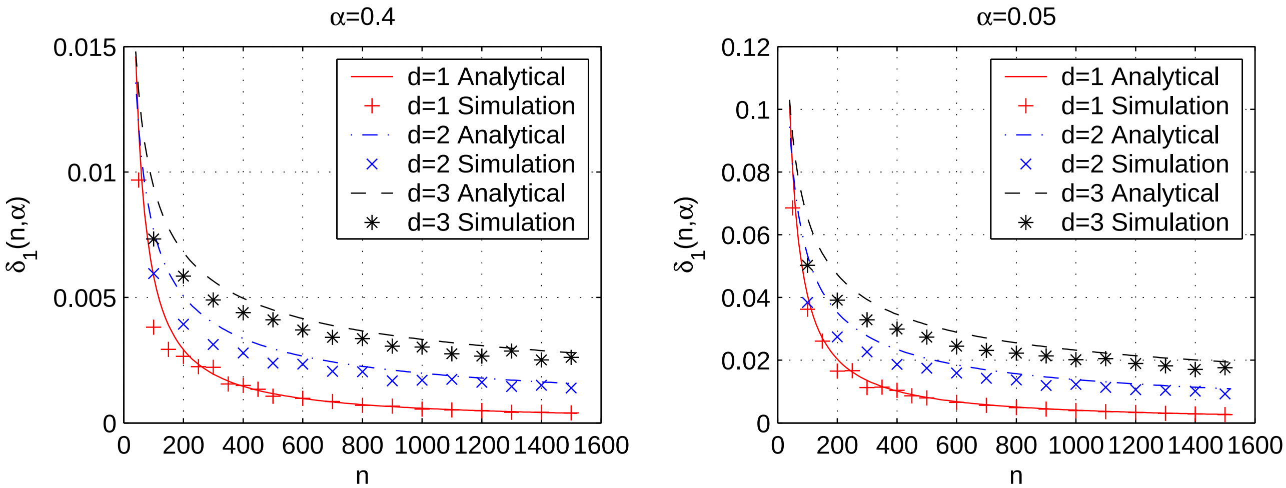


Fig. 10. Phase transition width of 1-connectivity *versus* the number of nodes n in d-dimensional space $d = 1, 2, 3$. The value of α is fixed in each scenario. For the analytical results, the boundary effect is ignored; for the simulation results, the boundary effect is eliminated by using the toroidal distance metric.

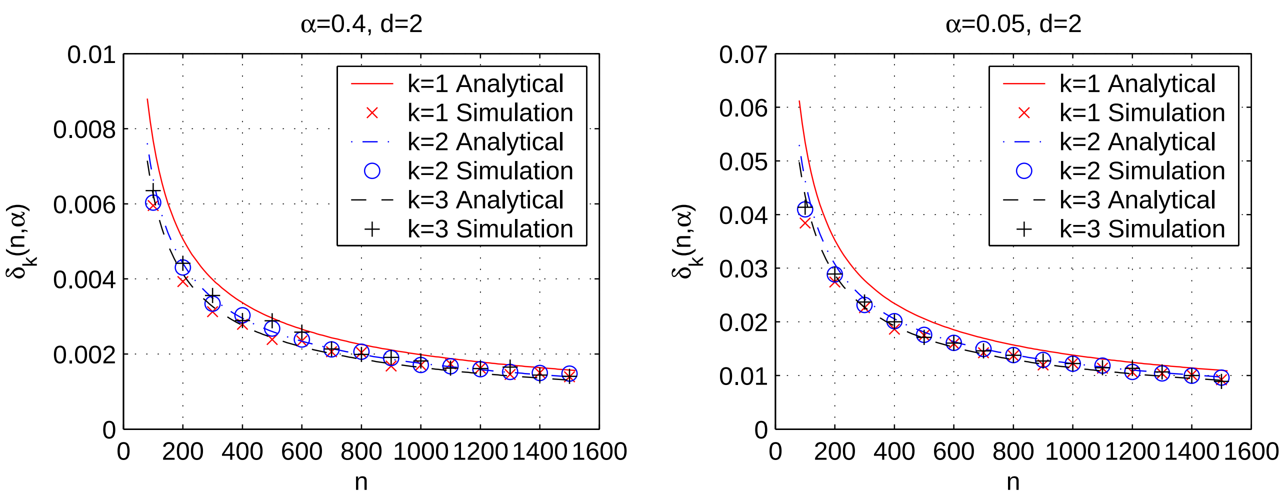
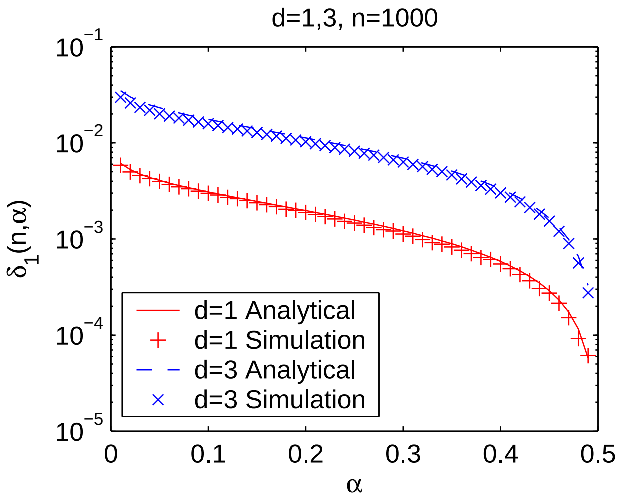


Fig. 11. Phase transition width of k-connectivity ($k = 1, 2, 3$) *versus* the number of nodes n in 2-dimensional space. The value of α is fixed in each scenario. For the analytical results, the boundary effect is ignored; for the simulation results, the boundary effect is eliminated by using the toroidal distance metric.

the transmission range to be just above the right boundary of the phase transition region so that the network achieves k-connectivity with a high probability while minimizing the energy utilization. For large n, we derived a generic analytical formula for calculating the phase transition width of k-connectivity for any fixed positive integer k in d-dimensional space ($d = 1, 2, 3$). We also derived an analytical formula for a modified version of the phase transition width in terms of the average node degree. To the best of our knowledge, our generic results have never been presented before. We also conduct simulations to verify our theoretical analysis. Our results were derived for large enough n, hence, they holds only when n is sufficiently large. When n is a small number (especially when n is comparable with $k!$), our results do not hold any more. Simulation results showed that n should be larger 200 when $k = 1$ and $d = 1$; and n should be larger than 600 when $k \leq 3$ and $d = 2,\ 3$. Our result also applies to mobile networks where nodes always move randomly and independently. These results are of practical value in the self-configuration of wireless multi-hop networks, and provide us useful design principles for wireless multi-hop networks as well.

In the theoretical analysis, we have ignored the boundary effect, which cannot be done in the real world. Hence, in the future, we will further investigate the phase transition width of k-connectivity considering the boundary effect. Furthermore, our theoretical results are derived for large n; sometimes, one will be also interested in the results for small values of n which may be encountered in real applications. Therefore, to investigate this problem for small values of n is another possible future work. We may also investigate the characterization of the phase transition width considering some more realistic channel models, e.g., considering shadowing [41] and interference [42].



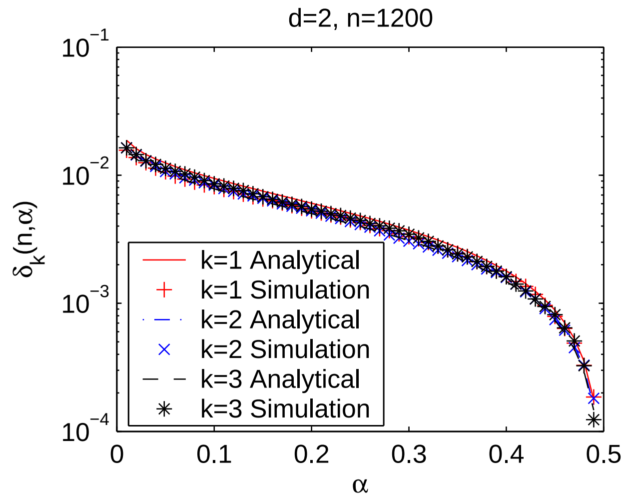


Fig. 12. Phase transition width of k-connectivity *versus* α in d-dimensional space $d = 1, 2, 3$. The value of n is fixed in each scenario. For the analytical results, the boundary effect is ignored; for the simulation results, the boundary effect is eliminated by using the toroidal distance metric.

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